Solutions, Math 240, Fall 2005, Final

1. **a.** If I pass this class, then I must have translated this sentence correctly.

b. If I translate this sentence incorrectly, then I will fail this class.

2.

1.	$\neg r \vee \neg f \rightarrow s \wedge \ell$	hypothesis
2.	$s \rightarrow t$	hypothesis
3.	$\neg t$	hypothesis
4.	$\neg s$	modus tollens: 2, 3
5.	$\neg s \vee \neg \ell$	addition: 4
6.	$\neg(s \wedge \ell)$	equivalence: 5 (DeMorgan's Law)
7.	$\neg(\neg r \vee \neg f)$	modus tolens: 1, 6
8.	$\neg\neg r \wedge \neg\neg f$	equivalence: 7 (DeMorgan's Law)
9.	$\neg\neg r$	simplification: 8
10.	r	equivalence: 9 (Double negation)

3. ($\Rightarrow$) If n is odd, then $n = 2k + 1$ for some integer k, and then n^2 is $(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Because $2k^2 + 2k$ is an integer, n^2 is odd.

($\Leftarrow$) We will prove the contrapositive. Suppose n is even — and so it is $2k$ for some integer k. Since n^2 is $(2k)^2 = 4k^2 = 2(2k^2)$, and $2k^2$ is an integer, we can conclude that n^2 is even.

4. Basis step ($n = 1$): $(x-1)(x^{1-1} + \cdots + 1) = (x-1)1 = x - 1 = x^1 - 1$.

Inductive step: Suppose that $(x-1)(x^{m-1} + x^{m-2} + \cdots + 1) = x^m - 1$. We want to demonstrate $(x-1)(x^{(m+1)-1} + x^{(m+1)-2} + \cdots + 1)$ is equal to $x^{m+1} - 1$.

$$\begin{aligned}
&(x-1)(x^{(m+1)-1} + x^{(m+1)-2} + \cdots + 1) \\
&= (x-1)(x^m + x^{m-1} + \cdots + 1) \\
&= (x-1)(x^m) + (x-1)(x^{m-1} + \cdots + 1) \quad && distribution \\
&= (x-1)(x^m) + x^m - 1 && inductive hypothesis \\
&= x^{m+1} - x^m + x^m - 1 \\
&= x^{m+1} - 1
\end{aligned}$$

5. $gcd(84, 60) = gcd(60, 24) = gcd(24, 12) = gcd(12, 0) = 12$

6. The flaw comes in choosing b: There may not be *any* elements related to a in R, in which case this choice of b cannot exist. (As an example, let R be a relation on $\{0, 1\}$ where $(1, 1)$ is the only pair in R. This is symmetric, and it is transitive; but it is not reflexive because $(0, 0) \notin R$.)

7. **a.** For the first digit, there are 8 choices; for the second, 2, and for the third, 9. Thus the total number of codes is $8 \cdot 2 \cdot 9 = 144$.

b. For the first digit, there are still 8 choices; but for the second digit, there are 9 choices, and for the third, 10, for a total number of $8 \cdot 9 \cdot 10 = 720$ possibilities. However, we must exclude the codes where the last two digits match; there is such a code for each combination of first and second/third digits, so we have $8 \cdot 9 = 72$ codes to exclude. Overall, then, there are $720 - 72 = 648$ possible area codes.

8. **a.** First I choose the three students receiving F's; then I can assign one of four grades to each of the remaining five students. Thus the total number of choices overall is

$$\binom{8}{3} 4^5 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} 2^{10} = 56 \cdot 1024 = 57{,}344 \,.$$

b. To count the number of choices where these two particular people receive F's, we must choose one of the remaining six students to receive the other F, and then there are five remaining choices to assign, for a total of

$$\binom{6}{1} 4^5 = 6 \cdot 2^{10} = 6{,}144 \,.$$

Overall, the probability would be

$$\frac{6145}{57344} = \frac{3}{28} \approx 10.7\% \,.$$

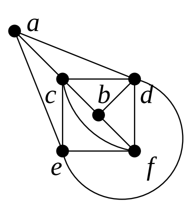
9. **a.** R is not reflexive, because 0 is an integer but $(0,0) \notin R$ (since $0^2 = 0 \not> 0$).

b. R is transitive: If $(a,b) \in R$, then both a and b have the same sign — they are either both positive or both negative. Similarly, if $(b,c) \in R$, then b and c share the same sign. Since a and c both have b's sign, they themselves share the same sign and so $(a,c) \in R$.

10. Both G and $\overline{G}$ can have an Euler circuit whenever the number of vertices is odd: For each odd n, we can use C_n as a graph with an obvious Euler circuit. We saw in class that every graph has an Euler circuit if all its vertices have even degree, so $\overline{C_n}$ has an Euler circuit. In $\overline{C_n}$, every vertex has degree $n-3$, which is even (since n is odd).

Suppose, however, that the number of vertices is even. We saw in class that if G has an Euler circuit, each vertex of G must have even degree. But if n is even, then each vertex of degree d (which would be even) would have degree $n-1-d$ in the complement. But $n-1-d$ is odd, and we know that a graph where some vertex has odd degree cannot have an Euler circuit.

11. **a.**



b. It is four: We can 4-color the graph by giving c, d, e, and f all different colors, giving a the same color as f and b the same color as e; in this coloring, the only pairs of vertices colored the same are a/f and b/e, and we can see that these vertices are not connected. However, we cannot 3-color the graph, as seen by the fact that $\{c,d,e,f\}$ have all pairs connected (it is K_4), and so these four vertices must all have different colors.

12.
```
(define (tree-sum data)
   (if (list? data)
       (if (null? data)
           0
           (+ (tree-sum (first data)) (tree-sum (rest data))))
       data))
```

13. (This space intentionally blank.)