Math 240, Fall 2005, Final

Name: ___________________________

1. [10 pts] Consider the following sentence.

I will not pass this class unless I translate this sentence correctly.

a. Translate the sentence into the form "if ____, then ____."

b. What is the contrapositive of your answer for part **a.**?

2. [15 pts] Prove on the following three hypotheses leads to the conclusion r, using inference rules and logical equivalences. Some possibly useful inference rules are listed at right.

1.	$(\neg r \vee \neg f) \rightarrow (s \wedge \ell)$	hypothesis
2.	$s \rightarrow t$	hypothesis
3.	$\neg t$	hypothesis

$$\frac{p \wedge q}{p} \quad \text{Simplification}$$

$$\frac{p}{p \vee q} \quad \text{Addition}$$

$$\frac{\begin{array}{l}\neg q \\ p \rightarrow q\end{array}}{\neg p} \quad \text{Modus tollens}$$

$$\frac{\begin{array}{l}p \vee q \\ \neg p\end{array}}{q} \quad \text{Disjunctive syllogism}$$

3. [20 pts] Prove: For every integer n, n is odd if and only if n^2 is odd. (Recall that we defined an integer to be *odd* if it is equal to $2k+1$ for some integer k.)

4. [20 pts] Prove using induction on n: For every integer $n \geq 1$ and for every real value x,

$$(x-1)(x^{n-1}+x^{n-2}+\cdots+1) = x^n - 1\ .$$

5. [8 pts] In Assignment 3, we saw the **Euclidean algorithm** for computing greatest common divisors: The GCD of a and 0 is a, while the GCD for a and a non-zero b can be computed by computing the GCD of $a \bmod b$ and b.

Use this algorithm to compute $gcd(84, 60)$; show your intermediate steps.

6. [6 pts] Identify the flaw in the following proof that every relation R that is both symmetric and transitive must be reflexive.

> To show that R is reflexive, we must show that for every element a in R's domain, $(a, a) \in R$. Let a be an arbitrarily chosen element in R's domain, and choose b be an element so that $(a, b) \in R$. Because R is symmetric, this means that $(b, a) \in R$. And because R is transitive, and we know $(a, b) \in R$ and $(b, a) \in R$, it must follow that $(a, a) \in R$.

7. [10 pts] U. S. area codes were introduced in 1947 and expanded in 1995.

a. In 1947, area codes were three-digit numbers where the first digit could not be 0 or 1, the second digit must be either 0 or 1, and the third digit could not be 0.* What is the maximum number of area codes allowed under this scheme?

b. By 1995, they had run out of area codes according to the above plan, and they allowed the second digit to be any digit between 0 and 8 (inclusive) and the third digit to be 0 — but they disallowed area codes where the last two digits match (such as 911, 800, and 888). (The first digit still cannot be 0 or 1.) How many area codes are allowed under this scheme?†

8. [10 pts] Suppose I have a class of eight students, and I want to assign each of them a letter grade (A, B, C, D, or F) where exactly three of the students receive an F.

a. How many ways can I choose to do this?

b. Suppose both you and your best friend (which is not yourself) are enrolled in my class, and suppose that I choose to assign letter grades completely randomly with the above restriction (3 F's). What is the chance that both you and your friend receive F's?

9. [10 pts] Suppose we define a relation R on integers where $(a, b) \in R$ whenever $ab > 0$.

a. Is R is reflexive or not? Prove your answer.

b. Is R is transitive or not? Prove your answer.

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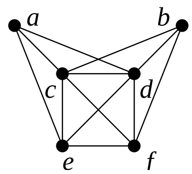
*The numbers were chosen based on the relative significance of the areas (by population): New York City got 212, the quickest to dial on rotary phones, while North Carolina got 704. Arkansas, with 501, fell in between.

†Experts predict that we will deplete the area code supply in 2030, at this point four-digit codes will likely be introduced.

10. [10 pts] Recall that an *Euler circuit* is a path that traverses every edge of a graph exactly once and returns to its starting point. For what integers n (beyond 2) can a graph G on n vertices and its complement $\overline{G}$ both have an Euler circuit? (An example of such a graph is C_5.) Prove your answer; you may use results proven in class.

11. [8 pts]

a. Is the following graph planar? Justify your answer.



b. What is the chromatic index of the above graph? Justify your answer.

12. [10 pts] Write a Scheme function `tree-sum` that will find the sum of all integers in a list, including sublists (and sublists of sublists, and sublists of them, ...). You may assume that the list contains only numbers and sublists. Some Scheme operations that you may find useful include `if`, `list?`, `null?`, `first`, and `rest`.

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(define (tree-sum data)
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13. [13 pts] Leave this question blank — or not; either way, you will receive full credit. *Have a joyous Christmas and New Year!*