Solutions, Math 240, Fall 2005, Exam 1

1. **a.** It's the contrapositive.

b.

p	q	$\neg q \rightarrow \neg p$
F	F	T
F	T	T
T	F	F
T	T	T

2. **a.** $S(Burch) \rightarrow K(Burch)$

b. $\exists x(C(x) \wedge K(x) \wedge \neg S(x))$

c. $\exists x \exists y(C(x) \wedge S(x) \wedge C(y) \wedge S(y) \wedge x \neq y)$

3.

1.	$r \wedge s \rightarrow t$	hypothesis
2.	$\neg t$	hypothesis
3.	r	hypothesis
4.	$\neg(r \wedge s)$	modus tollens: 2, 1
5.	$\neg r \vee \neg s$	DeMorgan's law: 4
6.	$\neg\neg r$	double negation: 3
7.	$\neg s$	disjunctive syllogism: 7, 5

4. **a.** $3 \equiv \mathbf{15} \pmod 6$

b. $9 \mid \mathbf{18}$

c. $25 \bmod \mathbf{12} = 1$

d. $gcd(24, \mathbf{20}) = 4$

5. $gcd(84, 60) = gcd(60, 24) = gcd(24, 12) = gcd(12, 0) = 12$

6. Because $d \mid a$, we can find some integer k so that $dk = a$. Note, then, that $(dk)c = ac$, or equivalently $(dc)k = ac$. Since k is an integer, we can then conclude that $dc \mid ac$.

7. Because n is composite, it must have a factor between 1 and n; call this a. Because $a \mid n$, there must exist an integer b so that $ab = n$. This b will also be positive, because a and n are positive. (If b were not positive, then ab would not be positive, but n is positive.) Also, b cannot be 1, because then ab would be a, which is not n as ab must be. Thus, a and b are both integers that are more than 1.

We claim that at least one of these is not more than $\sqrt{n}$. This is because, if they were both more than $\sqrt{n}$, then ab would be more than $(\sqrt{n})^2 = n$, but ab which is n, cannot be more than itself.

8. **a.** It is not one-to-one: For example, $f(3) = \lceil \sqrt{6} \rceil = 3$ and $f(4) = \lceil \sqrt{8} \rceil = 3$, but $3 \neq 4$.

b. It is not onto, because there is no x which maps to 1: We start with $f(0) = 0$, but then $f(1) = 2$, and the function is increasing, so it will never go down to 1 after it reaches 2.

9. We can place all of the positive rationals in a table such as the following, where we have a row for each possible denominator and a column for each possible numerator.

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Given this, we can select an ordering among the entries of the table, starting in the upper left corner and proceeding as indicated by the arrows. We skip over any numbers that equal numbers already encountered, though: For example, we skip over $2/2$ because we have already seen $1/1$.

i. $f(0) = 1/1$
ii. $f(4) = 1/3$
iii. $f(7) = 2/3$
iv. $f(10) = 1/5$

10. Basis step ($n = 1$):

$$\sum_{i=1}^{n}(2i-1) = (2 \cdot 1 - 1) = 1 = 1^2 = n^2$$

Inductive step: Suppose we already know that

$$\sum_{i=1}^{m}(2i-1) = m^2$$

for an integer m. We then compute the sum of the first m odd integers, which we want to show is $(m+1)^2$.

$$\sum_{i=1}^{m+1}(2i-1) = \left(\sum_{i=1}^{m}(2i-1)\right) + (2(m+1)-1) = m^2 + (2(m+1)-1) = m^2+2m+1 = (m+1)^2$$

We reached our goal of $(m+1)^2$, so the inductive step is complete.