Solutions, Math 240, Fall 2005, Exam 2

1. Basis step ($n = 0$):

$$\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2 - 1 = 2^{0+1} - 1\,.$$

Inductive step: Suppose we already know that

$$\sum_{i=0}^{m} 2^i = 2^{m+1} - 1$$

for an integer m. We want to show that the value of $\sum_{i=0}^{m+1} 2^i$ is $2^{m+2} - 1$. In the following sequence of equalities, we start by peeling off the final term of the summation, then applying the inductive hypothesis, and then simplifying.

$$\sum_{i=0}^{m+1} 2^i = \sum_{i=0}^{m} 2^i + 2^{m+1} = \bigl(2^{m+1} - 1\bigr) + 2^{m+1} = 2 \cdot 2^{m+1} - 1 = 2^{m+2} - 1\,.$$

Thus the first and last expressions are equal, which is what we wanted to complete the induction proof.

2. **a.**

n	s_n
0	1
1	0
2	1
3	1
4	2
5	3
6	5

b. The proof is wrong when it claims that the inductive hypothesis says that s_{m-1} is 0: When $m = 1$ (which is allowed since $m \geq 1$), then $m - 1$ is 0, which doesn't fall in the range for which the inductive hypothesis applies.

3. We first choose where to place the vowel (4 choices), then which vowel to place there (5 choices), then which consonant to place in each of the remaining spots (21 choices each), for a total of

$$4 \cdot 5 \cdot 21^3 = 185{,}220\,.$$

4. (i.) 4 is divisible by 2 or 3, because it is divisible by 2.
(ii.) 9 is divisible by 2 or 3, because it is divisible by 3.
(iii.) if b is divisible by 2 or 3, then $(a-1)b$, which is divisible by b, is also divisible by 2 or 3.
(iii.) if a is divisible by 2 or 3, then $a(b-1)$, which is divisible by a, is also divisible by 2 or 3.

5. There are ten different pairs of numbers that sum to 21: 1 and 20, 2 and 19, $\ldots$, 10 and 11. If we select eleven distinct numbers, each of which falls into one of these ten pairs, then according to the pigeonhole principle, at least one pair must have both of its numbers selected. The sum for this pair will be 21.

6. $2 \cdot \bigl(1 + 4 + \binom{4}{2}\bigr) \cdot (1+3) \cdot 2^3 = 2 \cdot 11 \cdot 4 \cdot 8 = 704$

7. Suppose we are counting the number of ways to choose two k-size subsets A and B from a set of n elements. One way to make this choice is to first choose the elements for A — there are $\binom{n}{k}$ ways of doing this — and then to select the element for B from the remaining $n-k$ elements — there are $\binom{n-k}{k}$ ways of doing this. According to the product rule, the number of ways for these choices is $\binom{n}{k}\binom{n-k}{k}$.

Another way, though, is to first select the $2k$ elements appearing in both A and B, and then select which k of them should go into A. In this case, we have $\binom{n}{2k}$ choices for the first selection and $\binom{2k}{k}$ choices for the second, for a total of $\binom{n}{2k}\binom{2k}{k}$.

8. To count the total number of packings without any restrictions, we are counting the solutions to the equation $s_U + s_N + s_K = 10$ using nonnegative integers only; there are $\binom{12}{2}$ such solutions. We do not want to count, however, solutions using more than six blue socks; if we had seven blue socks, we would have three days to account for, and there are $\binom{5}{2}$ ways to fill those last three days. Similarly, there are $\binom{5}{2}$ ways using more than six brown socks, and $\binom{5}{2}$ ways using more than six black socks. Thus, the number of ways using at most six pair of any single color is

$$\binom{12}{2} - 3 \cdot \binom{5}{2} .$$

(We would be over-subtracting if there were a way of having more than six blue and six brown socks; but there aren't, because we are only packing for ten days.)

9. This is like counting all arrangements of the letters A,A,A,A,B,B,B,B,C,C,C,C: Each student corresponds to a position in the sequence, and the letter in that position dictates which group the student is a part of. There are

$$\frac{12!}{4!4!4!} = 34{,}650$$

different arrangements.

This overcounts, however: for example,

A,A,A,A,B,B,B,B,C,C,C,C

is counted separately from

B,B,B,B,C,C,C,C,A,A,A,A,

but really these two group assignments are indistinguishible. Each assignment is counted $3!$ different times (each permutation of A, B, and C yields another assignment), so the real answer is

$$\frac{12!}{3!4!4!4!} = 5{,}775$$

10. There are 2^4 sequences with three heads at the front, and 2^4 with three heads at the end, but also there are 2^1 that have three heads at both the front *and* the end; thus, there are $2^4 + 2^4 - 2^1$ sequences with three heads at the front and/or the end. The probability of encountering one of these sequences when we randomly select one of the 2^7 sequences is

$$\frac{2^4 + 2^4 - 2^1}{2^7} = \frac{16 + 16 - 2}{128} = \frac{30}{128} .$$