Math 240, Fall 2005, Exam 2

Name: ____________________________

1. [10 pts] Prove using induction that for any nonnegative integer n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \, .$$

2. [10 pts] Suppose we define the following sequence.

$$s_n = \begin{cases} 1 - n & \text{if } n \leq 1 \\ s_{n-1} + s_{n-2} & \text{if } n > 1 \end{cases}$$

a. Compute the first seven values of this sequence — i.e., s_0 through s_6.

b. Consider the following "proof" regarding this sequence.

> **Proposition:** s_n is 0 for all $n \geq 1$.
>
> **Proof:** Basis step ($n = 1$): s_1 is $1 - 1 = 0$ by definition.
>
> Inductive step: For a given $m \geq 1$, suppose that $s_k = 0$ for all k between 1 and m (inclusive). We want to show that $s_{m+1} = 0$ also. According to the definition of s_n, we know $s_{m+1} = s_m + s_{m-1}$. According to the inductive hypothesis, both s_m and s_{m-1} are 0. Thus, s_{m+1} is $s_m + s_{m-1} = 0 + 0 = 0$, which is what we wanted to show.

Explain the exact point at which this "proof" makes an invalid step.

3. [10 pts] How many four-letter words contain one vowel (*a*, *e*, *i*, *o*, *u*) and three consonants?

4. [10 pts] Consider the following recursive definition of a set of integers. (**Note:** In class we have usually dealt with sets of strings. Here, we are dealing with integers, so the operations applied to them are multiplication and subtraction, *not* string concatenation.)

(i.) $4 \in S$
(ii.) $9 \in S$
(iii.) if a and b are in S, then so is $(a-1)b$.
(iv.) if a and b are in S, then so is $a(b-1)$.

Using structural induction, prove that all integers in S are divisible by either 2 or 3.

5. [10 pts] Using the pigeonhole principle, show that any set of eleven distinct integers between 1 and 20 must include two that sum to 21.

6. [10 pts] A deli has on its menu:

2 types of bread	choose one
4 types of cheese	choose up to two
3 types of meat	choose none or one
3 types of dressing	choose any number (including zero)

How many choices of sandwich does it offer?

7. [10 pts] Prove $\binom{n}{k}\binom{n-k}{k} = \binom{n}{2k}\binom{2k}{k}$ using a combinatorial argument.

8. [10 pts] How many ways can I pack blue, brown, and black socks for a ten-day vacation, using at most six (pair) of any single color? (Hint: How many ways are there to pack for a ten-day vacation without any restrictions? How many are there using more than six blue socks?)

9. [10 pts] Suppose I have a class of twelve students and I want to assign them into three groups of four each. How many ways are there for me to do this?

10. [10 pts] What is the probability of flipping a coin seven times and getting at least three heads in a row at the beginning and/or the end of the sequence of flips?