Take-Home Test 3, Math 240, Fall 2005

Due: 2:45pm, November 8. Value: 50 pts.

Instructions: Choose five of the following six problems to complete. If you submit solutions for all six, you will receive grades only for the first five that you answer.

In contrast to assignments, you must not discuss this take-home test with others, and you may not use electronic resources. The only written resources you may use are your textbook and class notes that you have written yourself. Be careful not to place your test where others might read your solutions. You may discuss the take-home test with me, although I won't promise to be helpful.

You will have the opportunity to complete a "rewrite," due the class day following when I distribute your initial grades. You will receive a three-point penalty on the rewrite, and your overall grade will be the higher of your initial grade and your rewrite grade. Your first graded version must be submitted with your rewrite.

Because of the rewrite policy, you should be careful not to discuss the test with others or access related resources even *after* you have submitted your initial version.

Problem A. Prove *combinatorially* that the following equation holds for every positive integer n.

$$\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}$$

Problem B. Suppose I enter a lottery where I choose six distinct integers from 1 to 55 (inclusive) and to win, my numbers must match at least four of the six integers in the winning set. What is the probability that I will win? Justify your answer.

Problem C. Prove or disprove: If R and S are both transitive relations on the same set A, then $R \cup S$ will also be transitive.

Problem D. Given a relation R on a set S, and given a subset $T \subseteq S$, we define $b \in S$ to be a *bound* of the set T if $t \, R \, b$ for all $t \in T$. Note that b is not necessarily in T.

A relation R on a set S is defined to be *linearly ordered* if it is reflexive, antisymmetric, transitive, and for all a and b in S, either $a \, R \, b$ or $b \, R \, a$ (or both — but because R must be antisymmetric, this only occurs when $a = b$).

Prove: For every linearly ordered relation R, every finite subset T of R's domain contains a bound of itself.

(Incidentally, this is not true for infinite subsets, even for subsets that have a bound: If we are using the $>$ relation on the set of rational numbers, then the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ has 0 as a bound, but the set does not itself contain any of its bounds.)

Problem E. Suppose we define a relation R on the positive integers where $a \, R \, b$ if ab is a perfect square (i.e., the square of some integer). Prove that R is an equivalence relation.

Problem F. Two vertices u and v are said to be *connected* if there is a path between them — i.e., there is a sequence of vertices $w_1, w_2, \ldots, w_k$ (for some integer k) where $w_1 = u$ and $w_k = v$, and where (w_1, w_2), (w_2, w_3), …, (w_{k-1}, w_k) are all edges in the graph. A graph is said to be *connected* if all pairs of vertices within it are connected.

Prove that for every positive even integer n, every graph where every vertex has degree at least $\frac{n}{2}$ must be connected. (Hint: Use the pigeonhole principle.)

Prove also that for every positive even integer n, there is an unconnected graph where every vertex has degree at least $\frac{n}{2} - 1$.